Lesson 1: Basic issues

1.1 What is game theory?

Game theory is interactive (interpersonal) decision theory.
Basic paradigms are:

- (one or) more decision makers (DM; usually, human beings)
- the DMs can influence the outcome. No one has a complete control of the situation.
- the DMs have (transitive) preferences w.r.t. the outcomes

Important specifications (not always valid, but valid in the basic model, the core model):

- the model is common knowledge (CK) among the players
- players have unbounded intelligence (reasoning and calculation ability)

Variants.

Players can be:

- not human (animals, plants, automata ...)
- single individuals or collectivities (remark: where is the identity of a human being? It is a collection of cells. Who decides? Depends also on mood, there are variations as time elapses)

1.2 Origins of game theory

The beginning is with the book by von Neumann and Morgenstern: *"Game theory and economic behavior"*, 1944.
Main goal is to explain (describe, understand) economic behavior, with a new piece of mathematics: *game theory*.
Behind this are:

- utilitarianism: utility-driven explanation of economics (the goals of the consumer, the individual)
- walrasian approach: competition (a "parametric" problem for the individual, being given the prices)
- technical issue: the former results by von Neumann, mainly the existence of saddle points for zero-sum games, but also his growth model.

What makes interesting GT is the fact that this theory applies (or should apply) to all of the (social) situations with more DMs involved.
As said by Aumann, in the New Palgrave Dictionary:

> ... a *unified field theory* for the *rational side* of *social sciences*

Where is rooted GT? In the prevailing economic theory (marginalist, neoclassical economics). It assumes that an *atomistic* explanation for social interactions is possible.

1.3 The main formal models

There is not a unique model.
There are three main models available:

- games in strategic (or normal) form
- games in extensive form
- games in characteristic (or coalitional) form

The first two are usually employed for non-cooperative games.
The last one is for cooperative games.

1.4 Games in strategic form

This is the main formal model.
It can be seen as made by two parts: the *game form* and the *preferences* of the players.

Briefly and informally, the *game form* is based on the following ingredients:

- the set of "players" (we shall assume it to be a finite set, which is not mandatory. Interesting models use a infinite set of players, e.g. "large" games, for the so-called large economies).
- the set of the actions (better: strategies) available to the players
- a relation between the actions of the players and the result (the "physical" outcome)

The second block, *preferences*, means that:

- each of the players has his evaluation of the result; i.e., has his preferences on the results.

Putting all together, we get a game.

The formal model is as follows.
The game form:

- N, a finite set
- $(X_i)_{i\in N}$, a family of non empty sets; we shall denote $X = \Pi_{i\in N} X_i$
- E, a set
- $g : X \to E$, a function

Plus

- $(\sqsupseteq_i)_{i\in N}$, a family of total preorders (i.e.: reflexive, transitive and total relations) on E.

So, a strategic form game can be formally described as:

$$(N, (X_i)_{i\in N}, E, g, (\sqsupseteq_i)_{i\in N})$$

We can also use a more concise representation.
Namely, the relations $\sqsupseteq_i$ induce, via g, relations $\succeq_i$ on X. That is, for every $x', x'' \in X$:

$$x' \succeq_i x'' :\Leftrightarrow g(x') \sqsupseteq_i g(x'')$$

So, a game in strategic form is defined also as:

$$(N; (X_i)_{i\in N}, (\succeq_i)_{i\in N}),$$

where $\succeq_i$ are total preorders on X.

1.5 Interpretation and modeling via the strategic form. Towards a "solution"

Formally a game in strategic form is $(N, (X_i)_{i\in N}, (\succeq_i)_{i\in N})$: that is, a game is given as soon as one identifies these sets and relations. So, these are be essential data, the ingredients that one should abstract from a situation. Not always easy.
The strategies available to a player? Are they given? By what? The problem has not to do only with the standard "arbitrary" simplifications. There could be strategies that we (the modelers, the external observers) miss.
Preferences? Not easier to detect than the strategy set.
Anyway, assume that we can really have a reasonable model in our hands.
And then?
We are interested in finding a "solution" for this model.
We shall need some inference rule, some pattern of behavior. Briefly, we need some *behavioral assumption*.
This assumption should be hidden, somehow implicitly, in our model. Of course! If we say that some objects are the data of the problem, this means that we have in mind to do something with them!!!
The idea that is lying behind all of this is clearly the paradigm of the rational DM. That is, the DM will *choose* in *a set* the *most preferred* element.
So, we shall say first something about the single DM problem.

1.6 The single decision maker, and connections with games

As said, the problem faced by a DM is to *select* the *action* (an action, in case of ties) that *produces* the *most preferred outcome*.
of course, behind this are some assumptions on the DM:

- has a clear representation of actions, can distinguish among them
- has the ability of choosing (enforcing) an action

- has a clear understanding of the function g, that maps actions to outcomes
- has a clear (mental?) picture of his preferences

We have an analogy with games.
We can describe a single DM problem as $(X, E, g, \sqsupseteq)$, where:

- X, E are sets
- $g : X \to E$
- $\sqsupseteq$ is a total preorder on E.

Or, as we did for strategic form games, we can induce a total preorder on X via g and $\sqsupseteq$.
So, we can see a single DM problem as $(X, \succeq)$.

Notice that the problem for the DM could be a problem of choice under risk, or uncertainty.

Risk:
$x \in X \mapsto L(x) \in E = \Delta(\hat{E})$ lotteries on $\hat{E}$.
Here we assume that the DM has preferences over $\mathcal{L}$, the set of lotteries on $\hat{E}$.

Uncertainty:
$x \in X \mapsto h \in E = \{h : X \times S \to \hat{E}\}$, where S is the set of states of nature.
The DM has preferences on E.

Under uncertainty, given a decision $\overline{x}$, we get $h(\overline{x}, \cdot) : S \to \hat{E}$. That is, for any given state of nature $\overline{s}$ we shall have a well specified outcome $h(\overline{x}, \overline{s}) \in \hat{E}$.
Notice the analogy with a game.
The choice of a strategy $\overline{x} \in X$, gives to a function $g(\overline{x}, \cdot) : Y \to E$.
Here, given a choice of a strategy $\overline{y} \in Y$ from player II, we shall have the outcome $g(\overline{x}, \overline{y}) \in E$.

1.7 Comments on the general model used for single and multiple DM models

The way in which we have represented the DM problem is correct? We start with alternatives and preferences. And from these we derive the choice. But this order is correct? From where are coming the alternatives, and the preferences?
Possibly, the preferences are a result of previous experience (evolutionary pressure, previous decisions made, strategic interactions in the past). For example, one is induced to have preferences that he considers as realistic, given the past experience.
So, there is a dynamic process going on. The way in which we make the *cut* is the best one?
This kind of issues is not too relevant if we are interested in a "local" analysis.
But if we want to reach the level of GT as a unifying language, a basic theory, then one must ask himself whether the model is so good.
Otherwise said: our DM is or not a product of a socio-historical process?
If yes, maybe we are looking for the foundations of economic behavior, but we are not looking at the right place.

1.8 Nash equilibrium as a solution: its justifications and related problems

For more than one DM?
Assume we have two DMs. If DM I chooses x, the result depends also on y, the choice of II.
We noticed the analogy with decision making under uncertainty. But, here, we do not have a completely *factorized* problem.
In decision making under uncertainty, $x \in X$ and $s \in S$ are completely independent (if not, we are using the model in a non appropriate way).
If s describes, for example, the weather conditions, we should stress that they are completely independent from the action x. Not only this.
The weather is also *indifferent to the choice* of x done by the DM. And also *indifferent to any theory* the DM could have on the weather and on the reasons to choose x.
So, despite some formal analogy, we cannot treat the connection between x (action chosen by DM I) and y (action chosen by DM II) in the same way as we treat the connection between x (action chosen by DM) and s (true state of nature).
The standard approach is to try to find a theory which is *as consistent as possible* with the single DM theory.
Let us add some further specifications on the links between the formal model we use (game in strategic form) and the concrete problem under exam.
The players choose their strategies *indipendently* and *contemporarily* (con-

temporaneity is to be understood from an informational point of view, not as strictly timing).
Players *do not have* any possibility of *communication* (so, no previous agreements are possible, w.r.t. the choices to be made).
Assuming all of this, do we have a solution *good for any season*? The answer is negative.
There is, however, a solution which occupies a distinguished position: Nash equilibrium.
The main reasoning behind this idea is based on the requirement that the theory should not be self-defeating.
A necessary condition for that can be described as follows.
Assume that the theory proposes a *unique* couple $(\overline{x}, \overline{y})$ as a solution (to be meant that $\overline{x}$ should be the choice for player I and $\overline{y}$ for player II).
If, e.g. player I has another action x s.t. $(x, \overline{y}) \succ_I (\overline{x}, \overline{y})$, then the theory is self defeating: player I will have an incentive not to *obey* the prescription of the theory.
Notice that the uniqueness assumption is a heavy one. Pure coordination games, or games like the battle of the sexes show how strong is this assumption.
Anyway, the reasoning above leads to the Nash equilibrium as a solution. A couple $(\overline{x}, \overline{y}) \in X \times Y$ is a Nash equilibrium for the game $(X, Y, \succeq_I, \succeq_{II})$ if

$$(\overline{x}, \overline{y}) \succeq_I (x, \overline{y}), \forall x \in X$$

$$(\overline{x}, \overline{y}) \succeq_{II} (\overline{x}, y), \forall y \in Y$$

There are other kinds of justification for the Nash equilibrium. But for them we have to drop the additional specifications that we introduced above. However, even if it is then conceivable for the players to make pre-play agreements, we shall assume that there is *no possibility* to make *binding agreements.*
A Nash equilibrium can be seen as incorporating the requirement about the *stability of a convention.* Of course, this means that the game has been played many times (or at least that this happened for similar games).
Clearly, there is something important which is missing in the picture. That is: from where does it come the convention? How it can be established? And, in case of multiple Nash equilibria, how it came that a specific one was selected?

Another kind of justification sees the Nsh equilibrium as a (stable?) steady state for some dynamic process. But ...
Which process?
Furthermore, there is a conflict between this interpretation and the assumption that players are intelligent (unlimitedly).
Some *try and error* process, or some kind of dynamic (myopic) adjustment

to previous plays can be considered. But it is sensible in situations where players are not so intelligent (*boundedly rational* human beings; animals; plants ...).

A final comment related with these last considerations. Sometimes it is said that the *core assumptions* of GT, about unlimited rationality and intelligence of the players, are not realistic. Notice that in any case it would be interesting and useful to have a solution for this ideal case.
If we do not have this, but we are obliged to work in the context of bounded rationality and intelligence, then we get other kinds of problems. We must add specifications, not easy to set. If it is possible to make mistakes, if the reasoning is not so deep, etc: *where* can be traced the border?