Lesson 2: Nash equilibrium

2.1 Representation of a game

A game in strategic form is usually represented as follows:

I\II	L	R
T	2 1	0 -1
B	1 0	-2 3

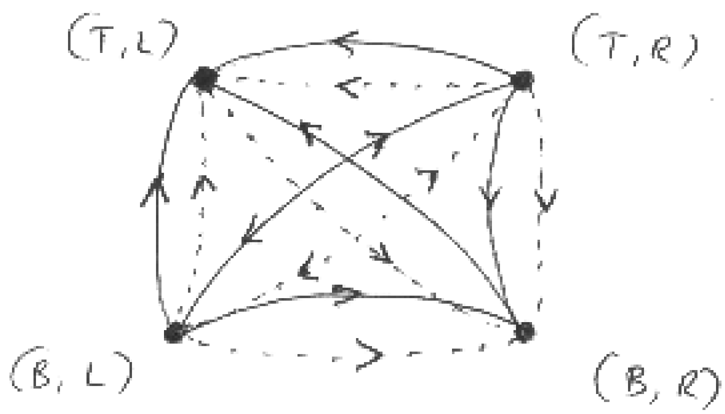
In the cells are the *values* of utility functions representing the preferences of players.
How to represent *directly* a game with $\succeq$, instead of using utility functions?
Three possible approaches:

- levels of grey
- representation of $\succeq$ on X as a subset of $X \times X$ (works fine if $X = \Re$, but difficult to extend to cases interesting for games)
- use of graphs (directed graphs, with arrows).

The last approach works with finite games. Moreover, if one is interested in Nash equilibria, a reduced form can be used. Let's just see a couple of examples:

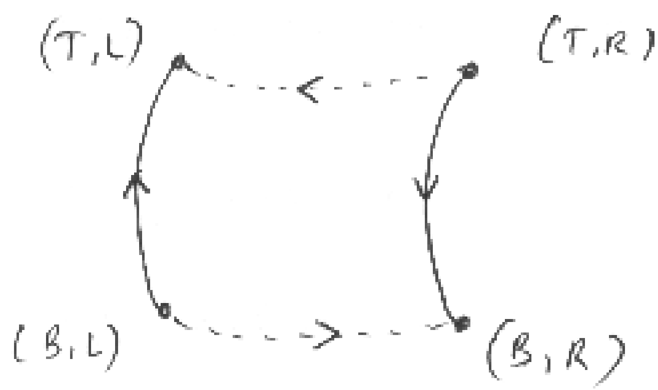
The battle of the sexes

I\II	L	R
T	2 1	0 0
B	0 0	1 2



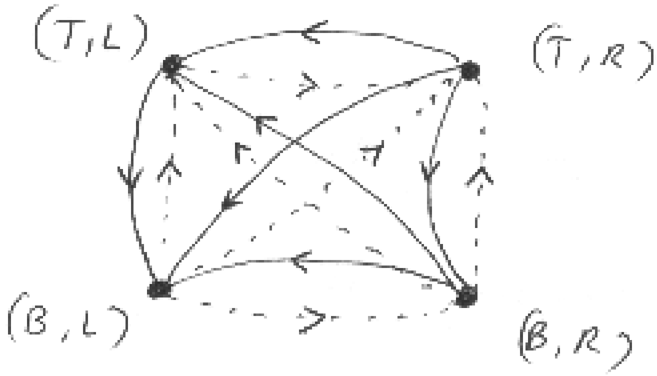
Arrows on a line mean that the vertex to which it points is weakly preferred to the other vertex. In the case of indifferece, there are two arrows, on both directions.
Solid lines refer to preferences of player I; the dotted ones to player II.

The reduced representation is:

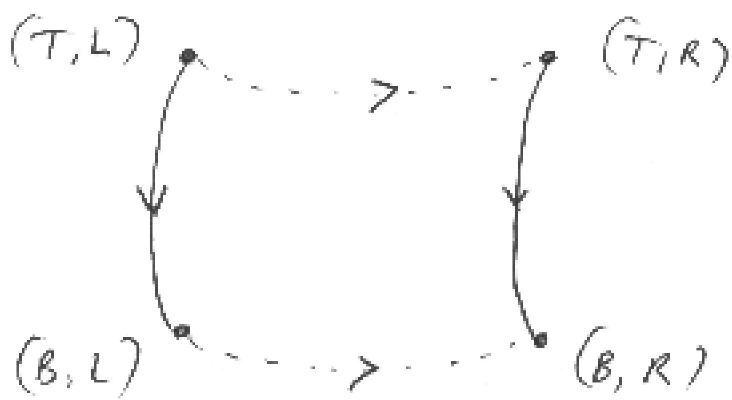


The prisoner's dilemma

I\II	L	R
T	3 3	0 5
B	5 0	1 1



Reduced representation:



2.2 Nash equilibrium: definition and discussion of some difficulties

I will give the formal definition just for two players.

So, instead of $(N,(X_i)_{i\in N},(\succeq_i)_{i\in N})$, we shall have $(\{I,II\},X,Y,\succeq_I,\succeq_{II})$. Or, shorter: $(X,Y,\succeq_I,\succeq_{II})$.

Definition: A couple $(\overline{x},\overline{y}) \in X\times Y$ is said to be a Nash equilibrium for the game $(X,Y,\succeq_I,\succeq_{II})$ if:
$(\overline{x},\overline{y}) \succeq_I (x,\overline{y}) \qquad \forall x\in X$
$(\overline{x},\overline{y}) \succeq_{II} (\overline{x},y) \qquad \forall y\in Y$

Let us see immediately some problems with Nash equilibria.

Example 1 (matching pennies) Consider the game:

I\II	L	R
T	1 -1	-1 1
B	-1 1	1 -1

It is immediate to check that this game does not have any Nash equilibrium.
Let's stress the fact that this difficulty seems to be an essential one. It is hard to imagine that this is a particularly wild game. Notice, furthermore, that the setting is a finite one (on infinite sets it is easy to give examples of optimization problems without solution; notice, however, that here we don't even have an approximate solution).

So, existence for Nash equilibrium is not guaranteed (but we shall see how to overcome this difficulty).
For uniqueness? Clearly, it is easy to provide examples of games with multiple Nash equilibria:

I\II	L	R
T	0 0	0 0
B	0 0	0 0

So, uniqueness is not guaranteed. Hence, in a game like this, we cannot make clear-cut predictions about the strategies chosen by the players. But who cares? All of the results are equivalent! We shall see that things, related with non uniqueness, can be much worse.

Example 2 (pure coordination game)

I\II	L	R
T	1 1	0 0
B	0 0	1 1

Here we have two Nash equilibria: (T, L) and (B, R). Notice that both players are indifferent about which one is chosen. Where are the troubles? The problem is that a Nash equilibrium is a *couple* of strategies. There is (in general) no "Nash strategy" for a player. How can players coordinate on one of the two Nash equilibria?

If we assume that players don't meet before the game, it is really difficult to imagine how a Nash equilibrium can be "played". If players are allowed to communicate before playing the game, it should not be difficult to coordinate on one of the two Nash equilibria.
The following example shows that pre-play communication will not erase all of the problems.

Example 3 (the battle of the sexes)

I\II	L	R
T	2 1	0 0
B	0 0	1 2

Here again we have two Nash equilibria: (T, L) and (B, R). Notice that I prefers (T, L) to (B, R), and that for II the opposite holds.

There is still an important example (the most famous game!), which shows an additional difficulty related with Nash equilibria.

Example 4 (prisoner's dilemma)

I\II	L	R
T	3 3	0 5
B	5 0	1 1

There is a unique Nash equilibrium, (B, R).
The outcome is inefficient, however. Both players strictly prefer the outcome from (T, L) to the equilibrium outcome.

2.3 Mixed extension of a finite game

We have seen some difficulties connected with the idea of Nash equilibrium.
We shall be able to overcome one of them, the non-existence problem.

The way to solve this problem is to embed the original game into a bigger one. More precisely, we shall extend the sets of strategies for the players, introducing the so-called mixed strategies.
Notice that this extension will not destroy previously existing Nash equilibria. Since, moreover, is not introducing anything revolutionary, the result will be that the other problems seen will remain (pure coordination game, battle of the sexes, prisoner's dilemma).
The classical approach to the mixed extension is to assume that players are von Neumann - Morgenstern decision makers.
That is, they have preferences on the set of lotteries that an be represented by vN-M utility functions.
Briefly, we assume that the preferences of each player on the relevant set of outcomes E can be represented by a function $u_i : E \to \Re$ s.t. for any couple of lotteries on E,

$L' = (p'_1, z'_1; \dots ; p'_m, z'_m)$
$L'' = (q''_1, z''_1; \dots ; q''_n, z''_n)$

we have:

$L' \succeq_i L'' \Leftrightarrow \sum_{k=1}^{m} p'_k u_i(z'_k) \succeq \sum_{k=1}^{n} q''_k u_i(z''_k)$

We can so build, starting from a game $(X, Y, \succeq_I, \succeq_{II})$, its so-called mixed extension $(\Delta(X), \Delta(Y), \succeq_I, \succeq_{II})$.
At least, this can be easily accomplished when X and Y are finite sets.
The interpretation is that a player, instead of choosing (deterministically) a strategy $x \in X$, chooses a mixed strategy.
That is, he fixes (deterministically) a lottery (i.e., probability distribution) on X. Then, to make the actual choice, he will use some device (random number generator?) that will follow the probability law decided by the player: the realization of this device will be the choice of a player.
In more detail, assume that $X = \{x_1, \dots , x_m\}$ and $Y = \{y_1, \dots , y_n\}$. Assume that $f, g : X \times Y \to \Re$ are von Neumann - Morgenstern utility functions which represent preferences of I and II respectively.
Assume that I chooses a "mixed strategy" (a probability distribution p on X, that is: $p = (p_1, \dots , p_m) \in \Delta(X)$) and analogously II chooses $q = (q_1, \dots , q_n) \in \Delta(Y)$, and assume that these choices *are made independently*, then we can evaluate the expected utility for both players:

$$f^{\Delta}(p, q) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j f(x_i, y_j)$$

$$g^{\Delta}(p, q) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j g(x_i, y_j)$$

Here $f^\Delta, g^\Delta : \Delta(X) \times \Delta(Y) \to \Re$ are the bilinear extensions of f, g from $X \times Y$ (that we can see as embedded into $\Delta(X) \times \Delta(Y)$) to $\Delta(X) \times \Delta(Y)$. In such a way we get a new game in strategic form, which is usually quoted as the mixed extension od the given game:

$$(\Delta(X), \Delta(Y), f^\Delta, g^\Delta)$$

We can suppress the suffix $^\Delta$, unless there is risk of confusion. So, we have:

$$(\Delta(X), \Delta(Y), f, g)$$