Lesson 4: Subjective and correlated equilibria

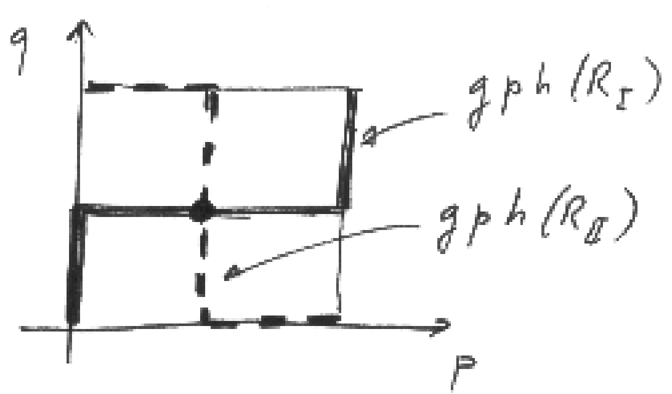
4.1 A couple of examples: matching pennies and the battle of the sexes

Matching pennies

I\II	L	R
T	$1,-1$	$-1,1$
B	$-1,1$	$1,-1$

Player 1: mixed strategy.
p_1 on T $\quad p_2$ on B $\quad (p_1+p_2=1 \quad p_1,p_2\geq 0)$
We can use just one variable, $p\in[0,1]$:
p on T $\quad 1-p$ on B
Analogously $\quad q$ on L $\quad (1-q)$ on R

Reduced graphs of the best reply correspondences



Remember: $R(p,q)=R_1(q)\times R_2(p)$
So, $(\overline{p},\overline{q})\in R(\overline{p},\overline{q})\Leftrightarrow$
$\Leftrightarrow (\overline{p}\in R_1(\overline{q})$ and $\overline{q}\in R_2(\overline{p}))\Leftrightarrow$
$\Leftrightarrow (\overline{q},\overline{p})\in gph(R_1)$ and $(\overline{p},\overline{q})\in gph(R_2)\Leftrightarrow$
$\Leftrightarrow (\overline{p},\overline{q})\in gph(R_1^{-1})$ and $(\overline{p},\overline{q})\in gph(R_2)\Leftrightarrow$
$\Leftrightarrow (\overline{p},\overline{q})\in gph(R_1)$ and $(\overline{p},\overline{q})\in gph(R_2)\Leftrightarrow$
$\Leftrightarrow (\overline{p},\overline{q})\in [gph(R_1)\cap gph(R_2)]$
So, looking for a fixed point for R is equivalent to look at a point in the

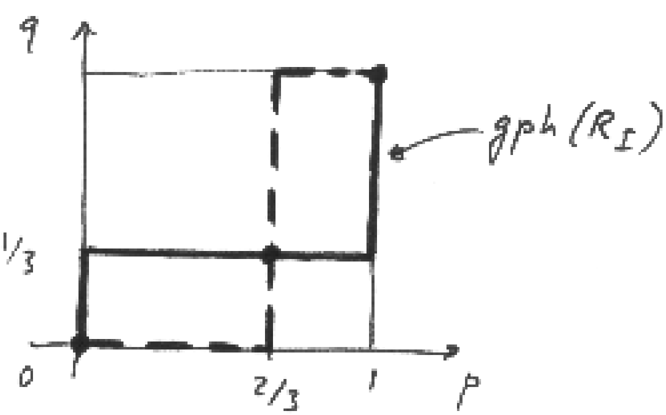
intersection of graphs of R_1 and R_2.

So, from the picture we see that the couple of mixed strategies $(\overline{p}, \overline{q}) = (\frac{1}{2}, \frac{1}{2})$ is the (unique) Nash Equilibrium for (the mixed extension of) matching pennies.

Battle of the sexes

I\II	L		R	
T	2	1	0	0
B	0	0	1	2

Same notations as before



So, three Nash equilibria in mixed strategies:
$\underbrace{(0,0)}_{(B,R)}, (\frac{2}{3}, \frac{1}{3}), \underbrace{(1,1)}_{(T,L)}$

Just a remark. Since I prefers (T, L) on (B, R), and viceversa for II, maybe the "new" mixed equilibrium could be a solution for this *conflict* between I and II.
The payoff is, however, $\frac{2}{3}$ for both.
That is, it is worse (for both) then the worst among the two pure Nash equilibria
So, the problem remains intact.
Why such a low payoff?
Because the probability distribution on the couples of pure strategies, resulting from the mixed equilibrium is the following:

		$\frac{1}{3}$	$\frac{2}{3}$
	I\II	L	R
$\frac{2}{3}$	T	$\frac{2}{9}$	$\frac{4}{9}$
$\frac{1}{3}$	B	$\frac{1}{9}$	$\frac{2}{9}$

So, there is a probability of $\frac{5}{9}$ that the result is 0 for both players.
Only with probability $\frac{4}{9}$ the players will get a positive payoff.
We shall come back on this, later.

4.2 Subjective equilibria

This is based on Aumann 1974.

Let's reconsider matching pennies.
Assume that there is an event E to which I assignes probability $\frac{2}{3}$ and II assignes probability $\frac{1}{3}$.
Assume that the players play the following strategies:
II plays always L.
I plays T if E, otherwise B.
So, the subjectively expected result is:
I $\rightarrow (T, L)$ with probability $\frac{2}{3}$, (B, L) with probability $\frac{1}{3}$
II $\rightarrow (T, L)$ with probability $\frac{1}{3}$, (B, L) with probability $\frac{2}{3}$.
Expected subjective payoff:
I $\rightarrow \frac{2}{3} \cdot 1 + \frac{1}{3} \cdot (-1) = \frac{1}{3}$
II $\rightarrow \frac{1}{3} \cdot (-1) + \frac{2}{3} \cdot 1 = \frac{1}{3}$
The game is not anymore competitive.

Nothing new: *horse races*!
Betting on events.
I pay you 1 if E occurs. You pay me 1 if E occurs.
Shall we make this *contract*?
Yes, if our subjective probability differ .... We both could expect to gain.
Of course, not both of us will gain!

Notice: that couple of strategies is not an equilibrium.
Clearly, given the strategy of II, it is better for I to switch to the (pure) strategy of playing T (always).
But with 3 players it can be achieved an equilibrium (subjective) !
Consider the game:

I\II	L			R		
T	0	8	0	3	3	3
B	1	1	1	0	0	0

I\II	L			R		
T	0	0	0	3	3	3
B	1	1	1	8	0	0

S D

$\nwarrow$ $\nearrow$

III

$$\text{Equilibrium:}\begin{cases} I & \rightarrow \quad B \\ II & \rightarrow \quad L \\ III & \text{any strategy} \end{cases}$$

Payoff: 1 for each player.

Let E be an event to which
I attributes probability (subjective) $\frac{3}{4}$
II attributes probability (subjective) $\frac{1}{4}$

Consider the strategy:
I plays T
II plays R
III plays S if E occurs, D otherwise.

What is interesting (can be easily checked) is that this is a subjective Nash equilibrium
That is, no one can profitably unilaterally deviate (taking into account the subjective probability they ascribe to E).
More interesting even is that: the result is *OBJECTIVELY* better for players and is $(3, 3, 3)$.

4.3 Correlated equilibria: the easy part of the story

We have seen, in the Battle of the Sexes:

I\II	L		R	
T	2	1	0	0
B	0	0	1	2

that mixed strategies give a low payoff because a lot of probability is "wasted" on bad payoffs:

		$\frac{1}{3}$	$\frac{2}{3}$
	I\II	L	R
$\frac{2}{3}$	T	$\frac{2}{9}$	$\frac{4}{9}$
$\frac{1}{3}$	B	$\frac{1}{9}$	$\frac{2}{9}$

If players can communicate before playing, they can come to an agreement to use correlated strategies.
That is, a coin is tossed: if heads, then I and II play T and L respectively; if tails, I plays B and II plays R.
Clearly, the expected utility for both players is strictly better than that obtained using the equilibrium mixed strategies. Now the joint probability distribution is different: nothing is "wasted" on bad cells.

I\II	L	R
T	$\frac{1}{2}$	0
B	0	$\frac{1}{2}$

Is there any possibility that this agreement is not violated by the players "a posteriori", i.e., when the result from the random device is known? It has to be a binding agreement to survive?
The answer is no. For the simple reason that both of the couples of strategies involved are Nash equilibria.
More generally, via correlation every payoff which is in the convex hull of the Nash equilibrium payoffs can be reached, without needing binding agreements.
But, as we shall see, we can do even better.

4.4 Correlated equilibria

We shall see a couple of examples. Both of them were provided by Aumann, 1974.

Ex 1 by Aumann (2 players)

I\ II	L		R	
T	6	6	2	7
B	7	2	0	0

This game has two pure strategies equilibria (B, L) and (T, R).
And a mixed equilibrium which gives a payoff of $\frac{14}{3}$ to both players.
Notice that in this case the expected payoff from the mixed equilibrium is not inefficient as in the case of the battle of sexes
Moreover, the expected payoff obtained by playing $\frac{1}{2}(B, L)$ and $\frac{1}{2}(T, R)$ is:

$\frac{9}{2}$ for both.
But $\frac{14}{3} = \frac{28}{6} > \frac{27}{6} = \frac{9}{2}$.
So, in this case, the idea that we used for the battle of the sexes does not give us a better result than the mixed strategy equilibrium.
Why all of this?
Obvious. The payoffs corresponding to (T, L) are good for the players. So that the probability that will fall on the corresponding outcome, instead of being *wasted* as in the battle of the sexes, on the contrary is *happily* put on that outcome (that players like).
Coming to numbers, while the expected payoff form $\frac{1}{2}(B, L)$ and $\frac{1}{2}(T, R)$ was $\frac{9}{2}$, the payoff for I and II is sensibly better in (T, L) giving 6 to both.
Of coure, playing (T, L) is not a feasible agreement, without binding agreements, since players have an incentive to deviate.
For the same reason, players cannot agree on a probability distribution which will put any positive probability on (T, L).
But all of this reasing is based on the assumption that both players know the true state of nature (after the randomizing device has given the result).

Consider the following mechanism.
A dice is thrown. According with the result, the following instructions are communicated (in a reliable way) to the players:

- if the outcome is 1 or 2, I is told to play T, while II is told to play L.
- if the outcome is 3 or 4, I is told to play T, while II is told to play R.
- if the outcome is 5 or 6, I is told to play B, while II is told to play L.

The expected payoff (for both players) is the following:

$\frac{1}{3} \cdot 6 + \frac{1}{3} \cdot 7 + \frac{1}{3} \cdot 2 = 5$

So, a better payoff than the mixed equilibrium payoff.
But players will follow the prescription?
The answer is *yes* (we mean: no incentive for unilateral deviation), due to the fact that they do not know the true state of nature, but just a signal which is only partially informative.
When player I is told to play B, he *knows* that player II is told to play L (trivial deduction from the mechanism).
But (B, L) is a Nash equlibrium, so no incentive for I to deviate.
If he told to play T, he *does not* know which strategy was supported to II.
He can only know that player II will play L with probability $\frac{1}{2}$ and R with probability $\frac{1}{2}$ (Bayes' rule).
So expected payoff for I is:

$$\frac{1}{2} \cdot \underbrace{6}_{(T,L)} + \frac{1}{2} \cdot \underbrace{2}_{(T,R)} = 4$$

If I deviates? No gain. His expected payoff is:

$$\frac{1}{2} \cdot \underbrace{7}_{(B,L)} + \frac{1}{2} \cdot \underbrace{0}_{(B,R)} = 3.5$$

Here is the second example.

Ex 2 by Aumann (3 players)

	L	R
T	0 0 3	0 0 0
B	1 0 0	0 0 0

S

	L	R
T	2 2 2	0 0 0
B	0 0 0	2 2 2

C

	L	R
T	0 0 0	0 0 0
B	0 1 0	0 0 3

D

Briefly: there are four Nash equilibria in pure strategies:
$(B, L, S) \to (1, 0, 0)$
$(T, R, S), (T, R, D) \to (0, 0, 0)$
$(B, L, D) \to (0, 1, 0)$
There are also mixed Nash equilibria. But no one of the resulting expected payoff coordinates is strictly greater than 1.

There is the following nice agreement:
III plays C
I,II toss a fair coin. *Without* informing III of the outcome.
If heads, they play (T, L); if tails, they play (B, R).
It is straightforward to check that this agreement gives payoff $(2, 2, 2)$.
And that there is no room for unilateral deviations. Of course, it is essential that III does not know the outcome of the money. Otherwise, it would be profitable for him to deviate.

4.5 Formal definition of correlated equilibria

There are essentially two (equivalent) ways to define correlated equilibria.
One explicitly refers to the mechanism and information available to players that originates the correlated equilibrium.
I used this when I described the case with two players.
The other one points directly to the resulting probability distribution on the Cartesian product of pure strategies.
I will follow the second path and I will give the definition in the case of finite strategy sets and two players only. Good references on Aumann 74 and 87, Osborne and Rubinstein 1994 and Myerson 1991. The last one emphasizes the viewpoint of mechanisms, incentive compatibility conditions and the revelation principle. His chapter 6 is really worth being read

So, let be given a finite game in strategic form with two players: (A, B, f, g)

Where $A = \{a_1, \ldots, a_m\}$ $B = \{b_1, \ldots, b_n\}$.
A probability measure on $A \times B$ is given as $\mu = (\mu_{ij})_{i=1,\ldots,m;j=1,\ldots,n}$.Where μ_{ij} means the probability assigned to the couple (a_i, b_j).
μ is a correlated equilibrium if

$$\text{for 1} \quad \begin{cases} \forall i_0 \in \{1, \ldots, m\}: \\ \sum_{j=1}^{n} \mu_{i_0,j} f(x_{i_0}, y_j) \geq \sum_{j=1}^{n} \mu_{i_0,j} f(x_i, y_j) \end{cases}$$

$$\text{for 2} \quad \begin{cases} \forall j_0 \in \{1, \ldots, n\}: \\ \sum_{i=1}^{m} \mu_{i,j_0} f(x_i, y_{j_0}) \geq \sum_{i=1}^{m} \mu_{i,j_0} f(x_i, y_{j_0}) \end{cases}$$

To understand the meaning of these conditions, consider (for 1) the situation in which $\sum_{j=1}^{n} \mu_{i_0,j} > 0$ (if this sum is $= 0$, the corresponding condition concerning i_0 is just that $0 \geq 0$).
We can then rewrite the condition as:

$$\sum_{j=1}^{n} \frac{\mu_{i_0,j}}{(\sum_{j=1}^{n} \mu_{i_0,j})} f(x_{i_0}, y_i) \geq \sum_{j=1}^{n} \frac{\mu_{i_0,j}}{(\sum_{j=1}^{n} \mu_{i_0,j})} f(x_i, y_j),$$

where the first term is the expected payoff for 1, when told to play x_{i_0}, calculated via bayesan updating of the prior, and assuming that 1 plays x_{i_0}, and the second one is similar to the first term, but assuming that 1 plays x_i.

As far as existence of correlated equilibria is concerned, it is immediate from the definition just given that a Nash equilibrium is also a correlated equilibrium.
Just μ is the product measure induced by the mixed strategies played by the players.
And, thanks to the independence assumption, bayesian updating is irrelevant.
It is left as an exercise to verify in detail the remark.
So, no question about the existence of correlated equilibria.